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Gamma generalized asymmetric curved normal distribution

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ABSTRACT
Modeling data situations that incorporate multiple modes presents challenges for both normal and skew-normal distributions. To overcome this challenge, we present a new class of asymmetric normal distributions designed to accurately represent both asymmetry and multimodality in datasets. Moreover, we explore the location-scale extension of this novel model and provide insights into parameter estimation using the maximum likelihood estimation method. To illustrate the practical utility of the model, we analyze a real-world dataset. Additionally, we perform a concise simulation study to showcase the effectiveness of maximum likelihood estimators in parameter estimation.

KEYWORDS
Model selection; Multimodality; Maximum likelihood estimation; Application; Generalized likelihood ratio test; Simulation.

1. Introduction

The normal distribution serves as a fundamental cornerstone in various statistical studies and holds a crucial position in probability theory, being indispensable for data processing and analysis. Despite its widespread utility, real-world data sets often exhibit asymmetry, deviating from the typical symmetry assumed by the normal distribution, which can complicate analysis. [2] introduced the "skew-normal distribution,"a mathematically tractable class of distributions that encompasses the normal distribution as a particular case. This distribution family is renowned for its efficacy in handling and understanding skewed data. [5] created a novel class of asymmetric normal distributions suitable for plurimodal datasets. They termed it the "extended skew curved normal distribution". The probability density function (p.d.f.) of the extended skew curved normal distribution is expressed as follows:

Let $\phi(\cdot)$ and $\Phi(\cdot)$ represent the probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of a standard normal variate, respectively. A random variable X is defined to adhere to the extended skew curved normal distribution (ESCND), denoted as ESCND(λ, β), where $\lambda \in \mathbb{R}$ and $\beta > -1$. The p.d.f $f(x; \lambda, \beta)$ for

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this distribution is expressed in the following form. For $x \in R$,

$$f(x; \lambda, \beta) = \frac{2}{\beta + 2} \phi(x) \left[1 + \beta \Phi \left(\frac{\lambda x}{\sqrt{1 + \lambda^2 x^2}} \right) \right]. \tag{1}$$

To enhance the flexibility in modeling plurimodal asymmetric distributions, this paper introduces a generalized form of the asymmetric normal distribution presented by [5]. We term this new distribution as the "gamma generalized asymmetric curved normal distribution (GGACND)."

The paper is organized as follows: In section 2, we introduce the definition and key properties of the GGACND. Section 3 presents the derivation of certain reliability measures, including the reliability function, failure rate, and mean residual life function, along with conditions for unimodal and plurimodal situations. In section 4, we propose a location-scale extension of the GGACND and outline its significant properties such as the characteristic function and reliability measures. Additionally, section 5 discusses the maximum likelihood estimation of the parameters of the extended GGACND, while section 6 explores a real-life application of the distribution.Furthermore, section 7 delves into the discussion of the generalized likelihood ratio test procedure to illustrate the significance of an additional parameter. In section 8, we conduct a brief simulation study to examine the performance of maximum likelihood estimators.

2. Gamma generalized asymmetric curved normal distribution

Here we define a generalized form of asymmetric normal distribution.

Definition 2.1. A random variable X is considered to follow a gamma generalized asymmetric curved normal distribution if its p.d.f is expressed in the following form, where $x \in \mathbb{R}$, and $\lambda, \beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma > 0$.

$$g(x; \lambda, \beta, \gamma) = \frac{\phi(x)}{\gamma + \beta} \left[\gamma + 2\beta \Phi(\theta(x)) \right] \tag{2}$$

Where $\theta(x) = \frac{\lambda x}{\sqrt{1+\lambda^2 x^2}}$, for convenience of notation.

We denote a distribution with the p.d.f (2) as GGACND(λ, β, γ).

Note that when

(1) $\gamma = 2$, GGACND(λ, β, γ) reduces to the extended skew curved normal distribution of [5].
(2) $\gamma = 0$, GGACND(λ, β, γ) reduces to the skew curved normal distribution of [1].
(3) $\beta = 0$, GGACND(λ, β, γ) reduces to the standard normal distribution .

For specific values of λ, β, and γ, the p.d.f given in (2) of GGACND(λ, β, γ) is plotted as in Figure 1.

The following findings revealed some structural features of GGACND(λ, β, γ).

Proposition 2.1. *If X follows GGACND(λ, β, γ), then $Z_1 = -X$ follows GGACND$(-\lambda, \beta, \gamma)$.*

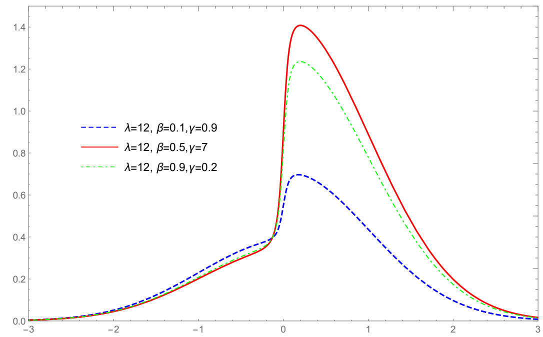

Figure 1. Probability plots of GGACND(λ, β, γ) for fixed values of λ and various values of β and γ

Proof. The p.d.f $g_1(z)$ of Z_1 is

$$\begin{aligned} g_1(z) &= g(-z; \lambda, \beta, \gamma)|\frac{dx}{dz}| \\ &= \frac{\phi(-z)}{\gamma+\beta}\left[\gamma + 2\beta\Phi(\theta(-z))\right] \\ &= g(z; -\lambda, \beta, \gamma), \end{aligned}$$

Since $\phi(.)$ is the p. d. f of standard normal variate. Hence Z_1 follows GGACND($-\lambda, \beta, \gamma$). □

Proposition 2.2. *If X follows GGACND(λ, β, γ), then $Z_2 = X^2$ follows a chi-square distribution with one degree of freedom.*

Proof. The p.d.f. $g_2(z)$ of $Z_2 = X^2$ is the following, for $z > 0$.

$$\begin{aligned} g_2(z) &= g(\sqrt{z}, \lambda, \beta, \gamma)|\frac{dx}{dz}| + g(-\sqrt{z}, \lambda, \beta, \gamma)|\frac{dx}{dz}| \\ &= \frac{\phi(-\sqrt{z})}{\gamma+\beta}\left[\gamma + 2\beta\Phi(\theta(-\sqrt{z}))\right]\frac{1}{2\sqrt{z}} + \\ &\quad \frac{\phi(\sqrt{z})}{\gamma+\beta}\left[\gamma + 2\beta\Phi(\theta(\sqrt{z}))\right]\frac{1}{2\sqrt{z}} \\ &= \frac{\phi(\sqrt{z})}{2(\gamma+\beta)\sqrt{z}}\left[2\gamma + 2\beta\left\{\Phi(\theta(-\sqrt{z}))\right.\right. \\ &\quad \left.\left. +\Phi(\theta(\sqrt{z}))\right\}\right] \\ &= \left(\frac{\phi(\sqrt{z})}{2\sqrt{z}}\right)\frac{1}{(\gamma+\beta)}\left[2\gamma+2\beta\right] \qquad (3) \\ &= \left(\frac{\phi(\sqrt{z})}{\sqrt{z}}\right) \end{aligned}$$

□

Proposition 2.3. *If X follows GGACND(λ, β, γ), then $Z_3 = |X|$ follows a standard half-normal distribution.*

Proof. For $x > 0$, the p.d.f of $g_3(z)$ of Z_3 is

$$\begin{aligned} g_3(z) &= g(z;\lambda,\beta,\gamma)|\frac{dx}{dz}| + g(-z;\lambda,\beta,\gamma)|\frac{dx}{dz}| \\ &= \frac{\phi(z)}{\gamma+\beta}\left[\gamma + 2\beta[\Phi(\theta(-z))\right] + \frac{\phi(-z)}{\gamma+\beta}\left[\gamma + 2\beta[\Phi(\theta(z))\right] \\ &= \frac{\phi(z)}{\gamma+\beta}\left[2\gamma + 2\beta\left\{\Phi(\theta(-z)) + \Phi(\theta(z))\right\}\right] \\ &= \frac{\phi(z)}{\gamma+\beta}\left[2\gamma + 2\beta\right]. \end{aligned} \tag{4}$$

□

The c.d.f of GGACND(λ, β, γ) with p.d.f (2) is obtained as follows.
For $x \in R$,

$$\begin{aligned} G(x) &= \int_{-\infty}^{x} g(t;\lambda,\beta,\gamma)dt \\ &= \frac{\gamma}{\gamma+\beta}\Phi(x) + \frac{2\beta}{\gamma+\beta}\left[\int_{-\infty}^{t}\phi(t)\Phi(\theta(t))dt\right] \\ &= \frac{\gamma}{\gamma+\beta}\Phi(x) + \frac{2\beta}{\gamma+\beta}\int_{-\infty}^{x}\int_{-\infty}^{\theta(t)}\phi(t)\phi(u)dudt \\ &= \Phi(x) - \frac{2\beta}{\gamma+\beta}\xi_0(x,\theta(t)), \end{aligned}$$

where

$$\xi_0(x,\theta(t)) = \int_{x}^{\infty}\int_{0}^{\theta(t)}\phi(t)\phi(u)\,dudt, \tag{5}$$

which can be computed using the software MATHEMATICA.

Next we derive the characteristic function of GGACND(λ, β, γ). To find the characteristic function, we need the following lemma proposed by [4].

Lemma 2.2. *For a standard normal random variable X with distribution function Φ we have the following for all $a, b \in R$*

$$E\left\{\Phi(aX+b)\right\} = \phi\left\{\frac{b}{\sqrt{1+a^2}}\right\}.$$

If X follows GGACND(λ, β, γ) with the p.d.f (2), then according to the definition of the characteristic function, we have the following expression for any $t \in \mathbb{R}$ and

$i = \sqrt{-1}$:

$$\begin{aligned}\psi_X(t) &= E(e^{itX}) \\ &= \frac{\gamma}{\gamma+\beta}\int_{-\infty}^{\infty} e^{itx}\phi(x)dx + \frac{2\beta}{\gamma+\beta}\int_{-\infty}^{\infty} e^{itx}\phi(x)\Phi(\theta(x))dx \\ &= \frac{e^{\frac{-t^2}{2}}}{\gamma+\beta}\left\{\gamma + 2\beta\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-(x-it)^2}{2}}\Phi(\theta(x))dx\right\}.\end{aligned} \tag{6}$$

On substituting $x - it = u$, in (6), we obtain

$$\psi_X(t) = \frac{e^{\frac{-t^2}{2}}}{\gamma+\beta}\left[\gamma + 2\beta\Phi(\theta(u+it))\right],$$

The propositions provide the expressions for both even and odd moments of $GGACND(\lambda, \beta, \gamma)$ as follows:

Proposition 2.4. *If X follows $GGACND(\lambda, \beta, \gamma)$ then for any k=1,2,...*

$$E(X^{2k}) = \frac{\gamma\Gamma(k+\frac{1}{2})2^k}{\sqrt{\pi}(\gamma+\beta)} + \frac{2\beta}{\gamma+\beta}\Lambda_{2k-1}, \tag{7}$$

where

$$\Lambda_{2k-1} = \int_0^{\infty} x^{2k-1}\phi(x)\Phi\left(\theta(x)\right)dx,$$

and can be readily evaluated using the software MATHEMATICA.

Proof. By the definition of raw moments,

$$E(X^{2k}) = \int_{-\infty}^{\infty} x^{2k} g(x;\lambda,\beta,\gamma)dx. \tag{8}$$

On substituting $x^2 = u$ in (8) to obtain,

$$\begin{aligned}E(X^{2k}) &= \frac{\gamma}{\gamma+\beta}\int_0^{\infty} u^k\phi(\sqrt{u})\frac{1}{\sqrt{u}}du + \frac{2\beta}{\gamma+\beta}\int_0^{\infty} u^k\phi(\sqrt{u})\Phi\left(\theta(x)(\sqrt{u}\right)\frac{1}{\sqrt{u}}du \\ &= \frac{\gamma}{\gamma+\beta}\int_0^{\infty} u^{k-\frac{1}{2}}\phi(\sqrt{u})du + \frac{2\beta}{\gamma+\beta}\int_0^{\infty} u^{k-\frac{1}{2}}\phi(\sqrt{u})\Phi\left(\theta(x)(\sqrt{u})\right)du,\end{aligned}$$

which leads to (7). □

Proposition 2.5. *If X follows $GGACND(\lambda, \beta, \gamma)$ then for any k=0,1,2,...*

$$E(X^{2k+1}) = \frac{\gamma(2^{k+\frac{1}{2}})}{(\gamma+\beta)\sqrt{2\pi}}\Gamma(k+1) + \frac{2\beta}{\gamma+\beta}\Lambda_{2k} \tag{9}$$

where

$$\Lambda_{2k} = \int_0^\infty x^{2k}\phi(x)\Phi\left(\theta(x)\right)dx,$$

which can be readily evaluated using the software MATHEMATICA.

Proof. By the definition of raw moments

$$E(X^{2k+1}) = \int_{-\infty}^{\infty} x^{2k+1} g(x;\lambda,\beta,\gamma)dx. \tag{10}$$

On substituting $x^2 = u$ in (10) we get,

$$\begin{aligned} E(u^{k+\frac{1}{2}}) &= \frac{\gamma}{\gamma+\beta}\int_0^\infty u^{k+\frac{1}{2}}\phi(\sqrt{u})\frac{1}{\sqrt{u}}du + \frac{2\beta}{\gamma+\beta}\int_0^\infty u^{k+\frac{1}{2}}\phi(\sqrt{u})\Phi\left(\theta(x)(\sqrt{u})\right)\frac{1}{\sqrt{u}}du \\ &= \frac{\gamma}{\gamma+\beta}\int_0^\infty u^{k}\phi(\sqrt{u})du + \frac{2\beta}{\gamma+\beta}\int_0^\infty u^{k}\phi(\sqrt{u})\Phi(\theta(x)(\sqrt{u}))du, \end{aligned}$$

which implies (9). □

3. Reliability measures and Mode

Here we explore certain properties of GGACND(λ,β,γ) with the probability density function (2), which are useful in reliability studies.

Let X follow GGACND(λ,β,γ) with the probability density function (2). From the definition of the reliability function $R(t)$, failure rate $r(t)$, and mean residual life function $\mu(t)$ of X, we derive the following results.

Proposition 3.1. *The reliability function $R(t)$ of X is defined as follows, where $\xi_0(t,\theta(x)) = \int_t^\infty \int_0^{\theta(x)} \phi(t)\phi(u)\,dudt$ is as defined in* (5).

$$R(t) = [1-\Phi(t)] + \frac{2\beta}{\gamma+\beta}\xi_0(t,\theta(x)).$$

Proposition 3.2. *The failure rate $r(t)$ of X is determined by the expression:*

$$r(t) = \frac{\phi(t)\left[\gamma + 2\beta\Phi(\theta(x)(t))\right]}{(\gamma+\beta)(1-\Phi(t)) + 2\beta\xi_0(t,\theta(x))}.$$

The failure rate plots of GGACND(λ,β,γ) for different values of γ are plotted given Figure 2.

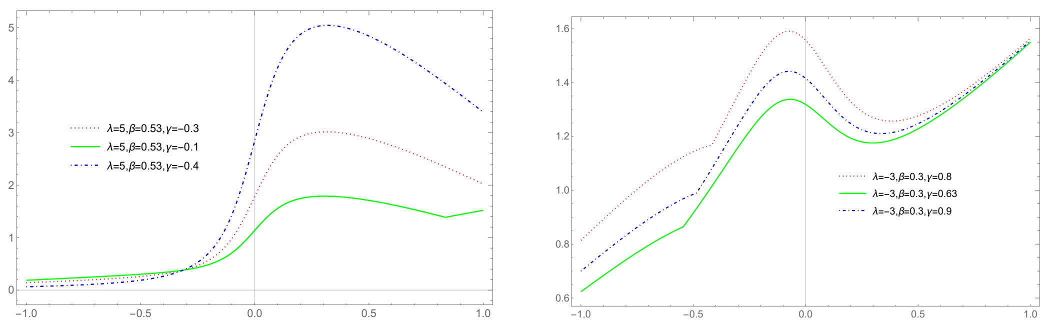

Figure 2. Failure rate plots of GGACND(λ, β, γ) for fixed values of λ, β and various values of γ.

From Figure 2, it is evident that for certain values of λ, β, and γ, the failure rate of GGACND(λ, β, γ) exhibits increasing, decreasing, and reversed 'S' shaped curves.

Proposition 3.3. *The mean residual life function of GGACND(λ, β, γ) is*

$$\mu(t) = \frac{1}{(\gamma+\beta)R(t)} \{\phi(t)\left(\gamma + 2\beta\Phi(\theta(x)(t))\right) + 2\beta\xi_0^*(t, \theta(x))\} - t, \tag{11}$$

where $\xi_0^*(t, \theta(x)) = \int_t^\infty \phi(x) \frac{d}{dx}(\int_0^{\theta(x)} \phi(u)du)dx$

Proof. By definition, the mean residual life function (MRLF) of X is given by

$$\begin{aligned} \mu(t) &= E(X - t/X > t) \\ &= E(X/X > t) - t, \end{aligned}$$

where

$$\begin{aligned} E(X/X > t) &= \frac{\gamma}{(\gamma+\beta)R(t)} \int_t^\infty x\phi(x)dx + \frac{2\beta}{(\gamma+\beta)R(t)} \int_t^\infty x\phi(x)\Phi(\theta(x))dx \\ &= \frac{1}{(\gamma+\beta)R(t)}[\gamma I_1 + 2\beta I_2], \end{aligned} \tag{12}$$

where

$$\begin{aligned} I_1 &= \int_t^\infty x\phi(x)dx \\ &= \phi(t) \end{aligned} \tag{13}$$

$$\begin{aligned} I_2 &= \int_t^\infty x\phi(x)\Phi(\theta(x))dx \\ &= -\int_t^\infty \phi'(x)\Phi(\theta(x))dx \\ &= \Phi(\theta(t))\phi(t) + \int_t^\infty \phi(x)\frac{d}{dx}\left(\int_0^{\theta(x)} \phi(u)du\right)dx \\ &= \Phi(\theta(t))\phi(t) + \xi_0^*(t, \theta(x)). \end{aligned} \tag{14}$$

By applying (13) and (14) in (12), we obtain (11). □

The following result established the criteria for $GGACND(\lambda, \beta, \gamma)$ is log concave.

Proposition 3.4. *Case 1: For $x > 0$, the p.d.f of $GGACND(\lambda, \beta, \gamma)$ is log concave if:*

(i) $\lambda < 0$, provided for all $\beta > 0$ and $\gamma > 0$ and
(ii) $\lambda > 0$, provided $|\frac{3\lambda^5 x^3}{(1+\lambda^2x^2)^{\frac{5}{2}}}| < |\frac{3\lambda^3 x}{(1+\lambda^2x^2)^{\frac{3}{2}}}|$

Case 2: For $x < 0$, the p.d.f of $GGACND(\lambda, \beta, \gamma)$ is log concave if:

(i) $\lambda > 0$, provided for all $\beta > 0$ and $\gamma > 0$ and
(i) $\lambda < 0$, provided $|\frac{3\lambda^5 x^3}{(1+\lambda^2x^2)^{\frac{5}{2}}}| < |\frac{3\lambda^3 x}{(1+\lambda^2x^2)^{\frac{3}{2}}}|$.

Proof. To establish that $ln[g(x; \lambda, \beta, \gamma)]$ is a concave function of x, it is sufficient to demonstrate that its second derivative is negative for all x. Then

$$\frac{d}{dx} ln[g(x; \lambda, \beta, \gamma)] = -x + \frac{2\beta\phi(\theta(x))(\theta^{'}(x))}{\gamma + 2\beta\Phi(\theta(x))}$$

and

$$\frac{d^2}{dx^2} ln[g(x; \lambda, \beta, \gamma)] = -1 - \Lambda_1 - \Lambda_2 + \Lambda_3$$

in which

$$\Lambda_1 = \frac{2\beta(\theta^{'}(x))^2\phi(\theta(x))\theta(x)}{\gamma + 2\beta\Phi(\theta(x))} \tag{15}$$

$$\Lambda_2 = \frac{4\beta^2(\phi(\theta(x)))^2(\theta^{'}(x))^2}{[\gamma + 2\beta\Phi(\theta(x))]^2} \tag{16}$$

and

$$\Lambda_3 = \frac{2\beta(\theta^{''}(x))\phi(\theta(x))}{\gamma + 2\beta\Phi(\theta(x))} \tag{17}$$

where

$$\theta(x) = \frac{\lambda x}{\sqrt{1 + \lambda^2 x^2}}$$

$$\theta^{'}(x) = \frac{\lambda}{\sqrt{1 + \lambda^2 x^2}} - \frac{\lambda^3 x^2}{(1 + \lambda^2 x^2)^{\frac{3}{2}}}$$

and

$$\theta^{''}(x) = \frac{3\lambda^5 x^3}{(1 + \lambda^2 x^2)^{\frac{5}{2}}} - \frac{3\lambda^3 x}{(1 + \lambda^2 x^2)^{\frac{3}{2}}}$$

Note that $\Lambda_1 > 0$ for $\beta > 0$ and $\theta(x) > 0$. And $\theta(x) > 0$ for all values of $\lambda > 0$. Consequently $\Lambda_2 > 0$ for all values of $\lambda, \beta, \gamma > 0$. Also $\Lambda_3 < 0$ for either $\beta < 0$ and $\theta(x)^{''}(x) > 0$ or $\beta > 0$ and $\theta(x)^{''}(x) < 0$. Hence (2) is log concave in these situations. □

As a consequence of Result 3.4, we can derive the following outcomes regarding the unimodality and multimodality of the $GGACND(\lambda, \beta, \gamma)$.

Proposition 3.5. *$GGACND(\lambda, \beta, \gamma)$ density is strongly unimodal under the following two cases.*
Case 1: For $x > 0$,

(i) if $\lambda < 0$, provided for all $\beta > 0$ and $\gamma > 0$ and
(ii) if $\lambda > 0$, provided $|\frac{3\lambda^5 x^3}{(1+\lambda^2x^2)^{\frac{5}{2}}}| < |\frac{3\lambda^3 x}{(1+\lambda^2x^2)^{\frac{3}{2}}}|$

Case 2: For $x < 0$,

(i) if $\lambda > 0$, provided for all $\beta > 0$ and $\gamma > 0$ and
(i) if $\lambda < 0$, provided $|\frac{3\lambda^5 x^3}{(1+\lambda^2x^2)^{\frac{5}{2}}}| < |\frac{3\lambda^3 x}{(1+\lambda^2x^2)^{\frac{3}{2}}}|$.

Remark 1. GGACND(λ, β, γ) density is multimodal under the following two cases.

Case 1: For $x > 0$,

(i) if $\lambda < 0$, provided for all $\beta < 0$ and $\gamma > 0$ and
(ii) if $\lambda > 0$, provided $|\frac{3\lambda^5 x^3}{(1+\lambda^2x^2)^{\frac{5}{2}}}| > |\frac{3\lambda^3 x}{(1+\lambda^2x^2)^{\frac{3}{2}}}|$

Case 2: For $x < 0$,

(i) if $\lambda > 0$, provided for all $\beta < 0$ and $\gamma > 0$ and
(i) if $\lambda < 0$, provided $|\frac{3\lambda^5 x^3}{(1+\lambda^2x^2)^{\frac{5}{2}}}| > |\frac{3\lambda\lambda^2 x}{(1+\lambda^2x^2)^{\frac{3}{2}}}|$.

4. Location scale extension

In this section, we explore an extended form of GGACND(λ, β, γ) by introducing the location parameter μ and scale parameter σ.

Definition 4.1. Let $X \sim GGACND(\lambda, \beta, \gamma)$ with the probability density function given in (2). Then $Y = \mu + \sigma X$ is said to have an extended GGACND with $\mu, \sigma, \lambda, \beta$, and γ with the following probability density function:

$$g(y, \mu, \sigma; \lambda, \beta, \gamma) \quad = \quad \frac{1}{\sigma(\gamma+\beta)}\phi\left(\frac{y-\mu}{\sigma}\right)[\gamma + 2\beta\Phi(\theta(y))], \tag{18}$$

where $\theta^*(y) = \frac{\lambda(y-\mu)}{\sigma^2+\lambda^2(y-\mu)^2}$, in which $y \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\sigma > 0$, and $\gamma \in \mathbb{R}$. A distribution with probability density function (18) is denoted as EGGACND$(\mu, \sigma; \lambda, \beta, \gamma)$. Clearly, when $\beta = 0$ or when $\lambda = 0$, EGGACND$(\mu, \sigma; \lambda, \beta, \gamma)$ reduces to $N(\mu, \sigma^2)$. Now we have the following results. The proof of these results is similar to the results given in $GGACND(\lambda, \beta, \gamma)$ and hence omitted.

Proposition 4.1. *The c.d.f $G(x)$ of $EGGACND(\mu, \sigma; \lambda, \beta, \gamma)$ with probability density function (18) is as follows, for $y \in \mathbb{R}$.*

$$G^*(y) = \Phi\left(\frac{y-\mu}{\sigma}\right) - \frac{2\beta}{\sigma(\gamma+\beta)}\xi_0^*(y, \theta^*(t)),$$

where $\xi_0^*(y, \lambda)$ *is as defined in* (5).

Proposition 4.2. *The characteristic function of EGGACND*$(\mu, \sigma; \lambda, \beta, \gamma)$ *is given by:*

$$\psi_Y(t) = \frac{1}{\sigma(\gamma + \beta)} e^{it\mu - \frac{t^2\sigma^2}{2}} \left\{\gamma + 2\beta\Phi\left(\theta^*(y)\right)\right\},$$

where $\theta^*(y) = \frac{\lambda(z+\sigma^2 it)}{\sqrt{\sigma^2+\lambda^2(z+\sigma^2 it)^2}}$.

Proposition 4.3. *The reliability function* $R(t)$ *of* Y *is defined as follows, where* $\xi_0(t, \theta(t)) = \int_t^\infty \int_0^{\theta^*(y)} \phi(\frac{y-\mu}{\sigma})\phi(v)dvdy$ *is as defined in* (5).

$$R^*(t) = \left[1 - \Phi\left(\frac{t-\mu}{\sigma}\right)\right] + \frac{2\beta}{\gamma + \beta}\xi_0(t, \theta^*(x))$$

Proposition 4.4. *The failure rate* $r^*(t)$ *of* Y *is given by:*

$$r^*(t) = \frac{\phi\left(\frac{t-\mu}{\sigma}\right)\left[\gamma + 2\beta\Phi\left(\theta^*(t)\right)\right]}{\left[1 - \Phi(\frac{t-\mu}{\sigma})\right](\gamma + \beta) + 2\beta\xi_0^*(t, \lambda)}.$$

5. Maximum likelihood estimation

The log-likelihood function, denoted as $\ln L$, for a random sample of size n drawn from a population that follows the EGGACND$(\mu, \sigma; \lambda, \beta, \gamma)$ distribution is provided as:

$$\begin{aligned} ln\, L &= c - nln(\gamma + \beta) - nln\sigma - \frac{1}{2}\sum_{i=1}^{n}\frac{(y_i - \mu)^2}{\sigma^2} \\ &\quad + \sum_{i=1}^{n} ln\left(\gamma + 2\beta\Phi(\theta^*(y))\right), \end{aligned} \tag{19}$$

where $c = -\frac{n}{2}ln\,(2\pi)$. Upon differentiation of (19) with respect to the parameters μ, σ, λ, β, and γ, and subsequently setting the derivatives equal to zero, we derive the following set of normal equations:

$$\sum_{i=1}^{n}\frac{(y_i - \mu)}{\sigma^2} + \sum_{i=1}^{n}\frac{2\beta\phi(\theta^*(y))\left(\frac{\lambda^3(y_i-\mu)^2}{[\sigma^2+\lambda^2(y_i-\mu)^2]^{\frac{3}{2}}} - \frac{\lambda}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}\right)}{\gamma + 2\beta\Phi(\theta^*(y))} = 0, \tag{20}$$

$$\frac{n}{\sigma} - \sum_{i=1}^{n}\frac{(y_i - \mu)^2}{\sigma^3} + \sum_{i=1}^{n}\frac{2\beta\lambda\phi(\theta^*(y))\left(\frac{(y_i-\mu)\sigma}{[\sigma^2+\lambda^2(y_i-\mu)^2]^{\frac{3}{2}}}\right)}{\gamma + 2\beta\Phi(\theta^*(y))} = 0, \tag{21}$$

$$2\beta\sum_{i=1}^{n}\frac{\phi(\theta^*(y))\left(\frac{(y_i-\mu)}{\sqrt{\sigma^2+\lambda^2(y_i-\mu)^2}}-\frac{\lambda^2(y_i-\mu)^3}{(\sigma^2+\lambda^2(y_i-\mu)^2)^{\frac{3}{2}}}\right)}{\gamma+2\beta\Phi\left(\theta^*(y)\right)}=0 \tag{22}$$

$$\sum_{i=1}^{n}\frac{2\Phi(\theta(x)^*(y))}{\gamma+2\beta\Phi(\theta^*(y))}-\frac{n}{\gamma+\beta}=0 \tag{23}$$

and

$$\sum_{i=1}^{n}\frac{1}{\gamma+2\beta\Phi(\theta^*(y))}-\frac{n}{\gamma+\beta}=0, \tag{24}$$

On solving the equations (20) to (24), we get the maximum likelihood estimate of the parameters of the EGGACND$(\mu,\sigma;\lambda,\beta,\gamma)$.

6. Applications

In this section, we explore a practical application of the EGGACND model using real-life data sourced from [3]. The dataset pertains to the daily milk production in kilograms for three milking times in cows.

We fitted the EGGACND$(\mu,\sigma;\lambda,\beta,\gamma)$ model to the dataset. To assess the model's suitability, we also applied the ESCND$(\mu,\sigma;\ \lambda,\ \beta)$ model proposed by [5] to the same dataset. We computed the Kolmogorov-Smirnov Statistic (KSS) values, Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC), and Corrected Akaike's Information Criterion (AICc) for both models. The numerical results are summarized in Table 1.

Table 1. Estimated values of the parameters for the model: EGSCND$(\mu,\sigma;\lambda,\beta)$ and EGGACND$(\mu,\sigma;\lambda,\beta,\gamma)$ with respective values of KSS, AIC, BIC and AICc.

Data set	Estimates of the parameters	ESCND$(\mu,\sigma;\lambda,\beta)$	EGGACND$(\mu,\sigma;\lambda,\beta,\gamma)$
1	$\hat{\mu}$	31.95	31.5
	$\hat{\sigma}$	4.62	4.46
	$\hat{\lambda}$	0.394	3.542
	$\hat{\beta}$	2.27	0.3
	$\hat{\gamma}$	-	20
	KSS	0.15315	0.0833026
	P-value	0.480675	0.981084
	AIC	173.539	170.963
	BIC	170.204	167.627
	AICc	176.267	173.69

Based on the results presented in Table 1, it is evident that the EGGACND$(\mu,\sigma;\lambda,\beta,\gamma)$ model offers a more suitable fit to the dataset examined in this study compared to the existing ESCND$(\mu,\sigma;\ \lambda,\beta)$ model. Consequently, the model discussed in this paper exhibits greater flexibility in terms of modeling perspectives. Additionally, we have depicted the histogram of dataset 1 alongside the corresponding fitted probability plots for ESCND and EGGACND in Figure 3. The figure illustrates that the EGGACND model provides a superior fit compared to the ESCND model for the dataset.

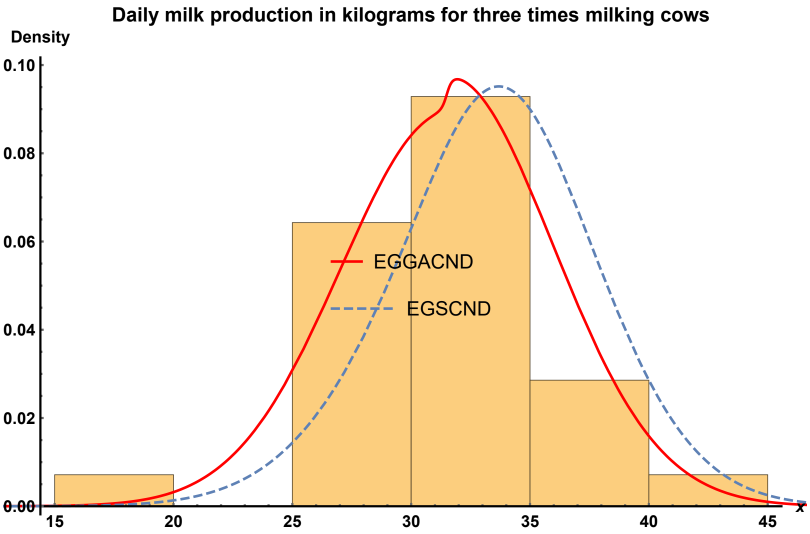

Figure 3. Histogram of Data set 1 and fitted distributions

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7. Generalized likelihood ratio test (GLRT)

In this section, we outline a test procedure for evaluating the parameter γ of the EGGACND model. To test the null hypothesis $H_0 : \gamma = 2$ against the alternative hypothesis $H_1 : \gamma \neq 2$ using the generalized likelihood ratio test (GLRT), the test statistic is defined as:

$$-2ln\lambda(x) = 2[lnL(\hat{\Theta}; x) - lnL(\hat{\Theta}^*; x)],$$

where $\hat{\Theta}$ represents the unrestricted maximum likelihood estimator of $\Theta = (\mu, \sigma; \lambda, \beta, \gamma)$, and $\hat{\Theta}^*$ is the maximum likelihood estimator of Θ when $\gamma = 2$. The test statistic provided is asymptotically distributed as χ^2 with 1 degree of freedom. For further elaboration, please refer to [6]. The results obtained using GLRT are summarized in Table 2.

Table 2. Likelihood values and GLRT test Statistic

	Data Set 1
$lnL(\hat{\Theta}^*; x)$	-83.473
$lnL(\hat{\Theta}; x)$	-79.863
GLRT Statistic	7.221

Since the critical value for the test with a significance level of 0.05 at one degree of freedom is 3.84, we reject the null hypothesis for the dataset under consideration. This rejection further reinforces the appropriateness of the EGGACND model for the dataset.

8. Simulation Study

To evaluate the performance of the maximum likelihood estimators for the parameters of the EGGACND$(\mu, \sigma; \lambda, \beta, \gamma)$, we conducted a brief simulation study. We generated observations using MATHEMATICA for the specified parameter sets: $\mu = 5$, $\sigma = 0.8$, $\lambda = 0.7$, $\beta = -0.5$, and $\gamma = -10$.

For comparison, we considered 500 bootstrap samples of sizes 30, 50, and 100 from the EGGACND distribution. We calculated the likelihood estimates of the parameters, average bias estimates, and average mean squared errors (MSEs) over 500 replications. The results are summarized in Table 3.

Table 3. Estimate of the parameters and corresponding bias and MSEs of *EGGACND* based on simulated data sets corresponding to parameter set $\mu = 5, \sigma = 0.8, \lambda = 0.7, \beta = -0.5$, and $\gamma = -10$.

Simulated Data Sets	Sample size	Parameter Set	Estimate	Bias	MSE
(1)	30	$\hat{\mu}$	5.6	0.6	0.36
		$\hat{\sigma}$	0.85	0.05	0.0025
		$\hat{\lambda}$	0.75	0.05	0.0025
		$\hat{\beta}$	-0.45	0.05	0.0025
		$\hat{\gamma}$	-9.5	0.5	0.25
	50	$\hat{\mu}$	5.5	0.5	0.25
		$\hat{\sigma}$	0.82	0.02	4e-04
		$\hat{\lambda}$	0.73	0.03	9e-04
		$\hat{\beta}$	-0.49	0.01	1e-04
		$\hat{\gamma}$	-9.7	0.3	0.09
	100	$\hat{\mu}$	5.1	0.1	0.01
		$\hat{\sigma}$	0.8001	1e-04	1e-08
		$\hat{\lambda}$	0.72	0.02	4e-04
		$\hat{\beta}$	-0.4902	0.0098	9.604e-05
		$\hat{\gamma}$	-10.098	-0.098	0.009604

From Table 3, it is evident that both the bias and MSE decrease as the sample size increases. This observation underscores the flexibility of the model discussed in this paper in terms of modeling.

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